THE MATHEMATICS TEACHERS' DEVELOPMENT SERIES

Pack 1: Investigative and Problem-Solving Approaches to Mathematics and their Assessment

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with Sue Burns, Jim Thorpe and Dylan Wiliam

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INTRODUCTION

1. Introductory Notes

'It is clear from the points made over and over again by witnessess that there is a large number of questions about the attainment of children which need more careful analysis than we have been able to give during our enquiry. These concern; the apparent lack of basic computational skills in many children; the increasing mathematical demands made on adults, the lack of qualified maths teachers, the multiplicity of syllabuses for old, new and mixed maths, the lack of communication between further and higher education, employers and schools about each group's needs and viewpoints, the inadequacy of information on job content or test results over a period of time, and the responsibility of teachers of mathematics and other subjects to equip children with the skills of numeracy.'

This extract from the report of the *Education, Arts and Home Office Sub-committee* of the *Parliamentary Expenditure Committee* published in July 1977, indicates that there was at that time widespread concern about the quality of maths teaching in schools. This sub-committee recommended that an enquiry be set up into the teaching of mathematics. In March 1978, the Government agreed, and established the *Committee of Inquiry into the Teaching of Mathematics in Schools* under the chairmanship of W H Cockcroft. The committee's report, *Mathematics Counts*, was published in 1982.

The impetus for the inquiry had been a perception that there was widespread dissatisfaction with the mathematical performance of school leavers. Many people saw a back-to-basics approach as the only solution to the problem. However, the report when published was of the opinion that such an approach would not work.

'. . . the results of a back-to-basics approach (as we understand it) are most unlikely to be those which its proponents wish to see, and we can in no way support or recommend an approach of this kind.'

Cockcroft Report, 1982

Instead what *was* recommended, in what has become the most widely quoted paragraph on the report, was that mathematics teaching at all levels should include opportunities for:

— exposition by the teacher;

— **discussion** between teacher and pupils and between the pupils themselves;

— appropriate **practical** work;

— consolidation and practice of fundamental skills and routines;

— **problem-solving**, including the application of mathematics to everyday situations;

— **investigational work**.

Two of these—exposition by the teacher, followed by consolidation and practice of fundamental skills and routines—have formed the staple diet of most students learning mathematics. The other four are more novel, and the last two in particular still pose difficulties for many teachers.

One of the major aims of incorporating open and practical ways of working is to reinstate the role of common sense in the maths classroom, to convince students that maths is intuitive rather than perverse, and that thinking, as well as remembering, is important.

These ways of working did not begin with the publication of the *Cockcroft Report*. Many teachers had already recognised their value and were using them in the classroom. In *Section Two: Assessment* we have used some early examples of schemes for the assessment of investigations as well as more recent ones.

The purpose of this Pack is to address these issues and to give support for their introduction, and assessment.

This *Introduction* deals with students discussion, investigational work and practical work.

Section One deals with the introduction of investigations and problem-solving into the classroom, and with the implications of this for the teacher's role.

Section Two deals with the assessment of investigations and problem-solving. These two ways of working will be called 'Open working'.

Section Three provides a bank of activities to use with students.

A word about words
The terms **investigation** and **problem-solving** have been used by different people to mean different things.

The *Cockcroft Report* appears to use the term *problem-solving* to apply only to practical or real-life applications of mathematics ('applied' mathematics), and *investigational* work is seen as belonging more to the 'pure' aspects of mathematics. The National Curriculum makes this distinction between applied and pure mathematics clear in the Attainment Targets 1 and 9; Problems are *'real-life problems'* pupils should *'investigate within mathematics itself'*

Some books and articles in the U K, written both before and after *The Cockcroft Report* use the term **mathematical problem-solving** to mean almost exactly the same as what the *Cockcroft Report* calls **investigational work.**

(See *Thinking things through* by Leone Burton.) This usage is more in line with what practising mathematicians would recognise; they tend to talk about problems rather than investigations. If the problem has not been solved it is described as open.

The meaning of **investigation**, as used in this Pack, is identical to its usage in the National Curriculum. We use **problem-solving** in a rather broader sense to include 'pure' problems as well as applied ones.

The most usual kinds of investigations can in fact be regarded as a particular kind of 'pure' problem where the hidden question is:

'Find a relationship between two or more variables in a particular mathematical situation.'

This question is rarely stated. Usually student material will simply present the mathematical situation and then say 'investigate'. But what is expected is that students should pursue pattern and order in the situation, that they should make explorations towards relationships between variables.

Chapter 1 deals with this kind of mathematical investigation. There is a situation to explore with some variables. Usually, but not always, there is an identifiable relationship between the variables and the process of investigating which moves the students towards the construction of that relationship.

2. How to use this pack

The materials in this pack can be used in the following ways:

— by heads of department for distribution as open learning and self-development exercises, either as part of a departmental strategy or to individuals experiencing difficulties;

— by groups, as team efforts, as the basis of departmental INSET;

— in departmental training sessions organised by a senior manager.

For your guidance exercises are coded as follows:

❃ individual work

❃❃ paired work

❃❃❃❃ group work

𝔻 whole department

Throughout the pack, material given to the student is called 'student material'.

3. Students' Discussion

'By the term "discussion" we mean more than the short questions and answers which arise during exposition by the teacher .[. . .] the ability to "say what you mean and mean what you say" should be one of the outcomes of good mathematics teaching. This ability develops as the result of opportunities to talk about mathematics, to explain and discuss results which have been obtained, and to test hypotheses.'

para 247, Cockcroft Report.

'Talking (describing; explaining; clarifying ideas; giving examples; making predictions; asking questions; reporting outcomes; talking through difficulties; discussing with peers).'

page B10, Non-Statutory Guidance, The National Curriculum Council, 1989.

Traditionally, the mathematics classroom has been a place where students sit and work quietly. As the quotes above illustrate there is a need for students to talk to the teacher and to one another about mathematics. The mathematics classroom should be somewhere mathematical discussion takes place. This can be encouraged by:

— asking students to work in pairs;

— giving them material or equipment to share, e.g. a computer;

— asking them to mark one another's work;

— asking them to set one another questions.

Having the seating arranged so that the students are all facing the teacher is not conductive to discussion between students. For this reason, it is recommended that you arrange the furniture in the classroom so that students are facing each other. This can be done by setting up group working areas for between four and eight students.

The best kinds of activities for encouraging students to talk about mathematics with one another are investigations, practical and problem-solving activities.

(How to get discussion amongst students started in the classroom is dealt with in the part on collaborative work in *Chapter 3*.)

4. Investigational Work

The term **investigational work** has, in many cases, been shortened to **investigations**. Although the term **investigations** includes most of what is intended by investigational work, some of the meaning has been lost. Investigations as such are dealt with in the first chapter. Here we wish to deal with investigational approaches to teaching items on a prescribed syllabus.

The Cockcroft Report saw investigational work as arising, at least in some cases, from the ordinary mathematics activity in the classroom. For example, in developing a ruler and compass construction to find the perpendicular bisector of a line, students may ask whether all four of the arcs have to have the same radius. Allowing the students to experiment by varying the arcs will lead to greater insight into how the method works.

The general principal here is sometimes to replace presentation of a piece of mathematics followed by exercises, with a more open approach which allows the students some flexibility. In order to make an introduction to a topic more open you can alter your approach so that students may:

— make decisions *en route*;

— construct their own examples;

— draw their own conclusions.

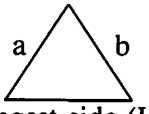
This is done within a tight framework, to facilitate the learning of the particular item of mathematics.

❃ Exercise

Read pages D5–D7 of the *National Curriculum (Mathematics): Non-Statutory Guidance,* 1989. You might like to add one more example to the table on page D7.
Another example of such a task is *Triangle Triples* which is a possible introduction to Pythagoras' Theorem. The students can be encouraged to share out the work of construction the triangles, and to use each other's data.
Once the table is completed, students quickly spot that two of the columns are equal when the triangle has a right angle. If they don't, a hint to look at the angles usually suffices.

Many students will go on to make other observations. Many will draw more triangles in order to investigate their conjectures. Some students will notice that where there is an acute angle opposite the longest side, $a^2 + b^2 > (L)^2$ and that where there is an obtuse angle opposite the longest side, $a^2 + b^2 < (L)^2$. This will be a useful basis for a later understanding of the cosine rule; $c^2 = a^2 + b^2 - 2abCosC$. CosC is positive for acute angles, zero for 90° and negative for obtuse angles. The difference between $(L)^2$ and $a^2 + b^2$ may be seen by some of the students as a measure of the 'acuteness' or 'obtuseness' of the angle concerned.

TRIANGLE TRIPLES



longest side (L)

- Construct the 12 triangles, one for each triple, using a ruler and compasses.
- Complete the table.
- What do you notice?

a	b	L	a^2	b^2	a^2+b^2	L^2	Notes about the triangle
5	12	13					
8	12	14					
6	10	15					
8	15	17					
15	16	19					
13	17	20					
14	14	22					
10	22	24					
7	24	25					
18	19	26					
20	22	28					
20	21	29					

5. Investigations and Problem-Solving

Following the publication of *The Cockcroft Report* many local education authorities and schools have tried to implement its recommendations. The introduction of the GCSE has followed hard on the heels of *The Cockcroft Report*. For a large number of maths teachers, this has meant grappling with coursework in mathematics. GCSE coursework is generally investigational or problem-solving work.

More recently, of course, the National Curriculum has arrived. This, too, emphasises the importance of investigational and problem-solving work. Two of the fourteen Attainment Targets deal specifically with these areas:

Attainment Target 1:

Pupils should use number, algebra and measures in practical tasks, in real-life problems, and to investigate within mathematics itself.

Attainment Target 9:

Pupils should use shape and space and handle data in practical tasks, in real-life problems, and to investigate within mathematics itself.

'Using and applying mathematics, as represented in Attainment Targets 1 and 9 and the associated elements of the programmes of study, should stretch across and permeate all other work in mathematics, providing both the means to, and the rationale for, the progressive development of knowledge, skills and understanding in mathematics.'

page D3, Mathematics Non-Statutory Guidance, National Curriculum Council, 1989.

A major aim of an investigation or problem-solving lesson is to get students to think for themselves. In doing this they will practise drawing on their repertoire of mathematics, which will help to consolidate what they can already do. They will also learn mathematical facts and techniques. These will depend partly on the students and partly on the activity.

These open ways of working make new demands on the teacher. In a 'chalk and talk' lesson the main aim is the transmission of a particular item of knowledge. The teacher is the possessor of the knowledge. The students are not. The aim of the lesson is for the teacher to put the knowledge across to as many students in the class as possible in as meaningful a way as possible.

The teacher's role is to act as a guide to the students, encouraging their progress, mainly through the use of leading questions, sometimes offering guidance and occasionally answering direct questions, and the first two chapters will deal with the major processes that students go through when they are working openly, and will provide the teacher with exercises and support for this way of working.

6.Practical Work

Practical work in mathematics can have two distinct meanings:

i) Making mathematics a more practical, hands-on subject, like science . This involves using **equipment**, especially 3D equipment.

ii) Making mathematics seem more relevant to the students through the **practical applications** of mathematics. This involves linking the mathematics you teach to applications in other subjects areas and outside the school.

Practical Equipment

The use of practical equipment in primary schools is now well established and owes much to the child-centred psychologies that were being followed in the nineteen-sixties. Dienes, Bruner and Piaget emphasised the benefits of keeping mathematics a concrete activity for the early stages of learning.

Most secondary school students are at one of the two stages of thinking named by Piaget 'concrete operational' and 'formal operational'. The onsetof the formal operational stage was marked by the development of the ability to 'reason hypothetically and independently' and Piaget believed that children acquired this formal operational thinking between the ages of 12 and 15.

Many people have interpreted this as meaning that children have no need for practical apparatus or concrete experience once they have acquired formal operational thinking, but this is not true. Piaget saw formal operational thinking as being the ability to 'reason hypothetically and independently *on concrete states of affairs'* (author's emphasis) and the ability to reason hypothetically about *abstract objects* as being at a higher level.

Bruner, Dienes and Piaget all express the view that concrete practical experience is appropriate for all students, whatever their stage of development.

When learning something new, there is always the need to relate it to something known about already. This often means using concrete experience. For instance, a teacher learning about databases may build what she is learning onto her experience of using a card index system.

The Cockcroft Report also emphasises this point:

'It is too often assumed that the need for practical activity ceases at the secondary stage but this is not the case. Nor is it the case that practical activity is needed only by those pupils whose ability is low: pupils of all levels of attainment can benefit from the opportunity for appropriate practical experience. [. . .] The results of the practical testing carried out by the Assessment of Performances Unit and described in the reports of both primary and secondary tests illustrate clearly the need to provide opportunities for practical experience and experiment for pupils of all ages.' *para 247, Cockcroft Report*

As an example, when introducing volume, students could make cuboids from centicubes. Counting the number of centicubes will give the volume of the cuboid. The multiplication rule for calculating the volume of a cuboid can be derived directly from this practical experience. The cuboids could also be drawn on isometric paper.

If there is no equipment at all available to you at present, it is suggested that you start with a small number of very basic items. It is assumed that graph paper, rulers, compasses, protractors and set squares are already available.

Basic Equipment List

- scissors
- glue
- card
- poster paper for display
- centicubes
- selection of 2D shapes and 3D solids
- angle indicators
- tape measure or steel rule
- cm squared paper
- isometric paper
- cm squared dotty paper

(cm squared paper, isometric paper, cm squared dotty paper: included in this pack, p.121)

- colours
- calculators
- metre rule
- tracing paper
- multi-link
- mirrors
- counters

❃ Exercise

The following activities involve the use of equipment, you will find details of them in *Section Three: Student Activities*, p.81. Choose one to try out with a class or small group of students.

- *Cone Code Worksheet* (p.84)
- *Estimating and Measuring Length* (p.88)
- *Lift Regulations* (p.92)
- *Measuring Small Thicknesses* (p.97)
- *Parallelogram Patterns* (p.101)
- *The Solar System* (p.105)
- *The Weather* (p.117)

Practical Applications

If students learn mathematics as an abstract set of rules and techniques they are not likely to become very skilled in its use. In order to be a good user of mathematics it is necessary to know *when* to use it. This requires teaching and practice. In general, no new topic in mathematics should be introduced without referring to its uses outside the mathematics classroom. Preferably this should include a use in another school subject and a use outside school.

Investigations and problem-solving activities, which are described in *Chapters 1* and *2*, will help students with practical applications and mathematical selection. Selecting mathematics is emphasised in Attainment Targets 1 and 9 of the National Curriculum.

When you are introducing a new topic, think carefully about its uses and make sure you bring them into your introduction.