SECTION ONE: INVESTIGATION AND PROBLEM-SOLVING

CHAPTER 1

INVESTIGATIONS

Introduction

The first thing to remember when you introduce an investigation to students is that you are **not** aiming to teach a particular **fact**.

Investigations are about processes. (See below.)

It is not possible to specify exactly what learning will take place during each investigation. What students learn will depend on:

— their current state of knowledge;
— how they respond to the initial stimulus;
— how familiar they are with investigative processes;
— the guidance you provide when they are doing the investigation.

For instance, in any one investigation lesson a student might learn any or all of the following:

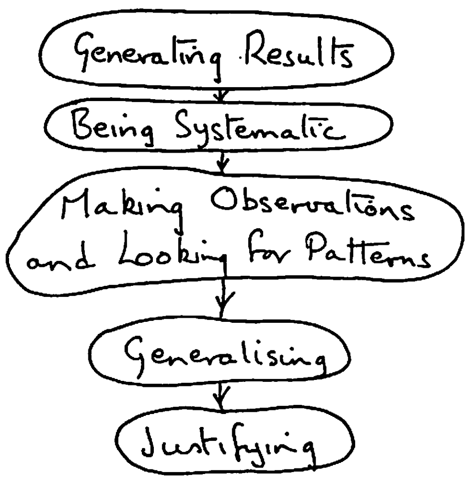
i) A new mathematical fact or technique. This will be something which the students need in order to pursue the investigation. The students will seek the new information themselves. They could find it out from you or from a peer
ii) A way of generating results systematically, e.g. holding a variable constant
iii) A way of organising results, e.g. into a table
iv) A way of looking for a pattern, e.g. finding differences in a sequence

There is a great deal for you to take in when you start to use investigations with your classes, and for that reason it is sensible to concentrate on one or two aspects at a time.

Don't try to do everything at once. Remember that if your students are new to investigating they will not know what is expected of them.

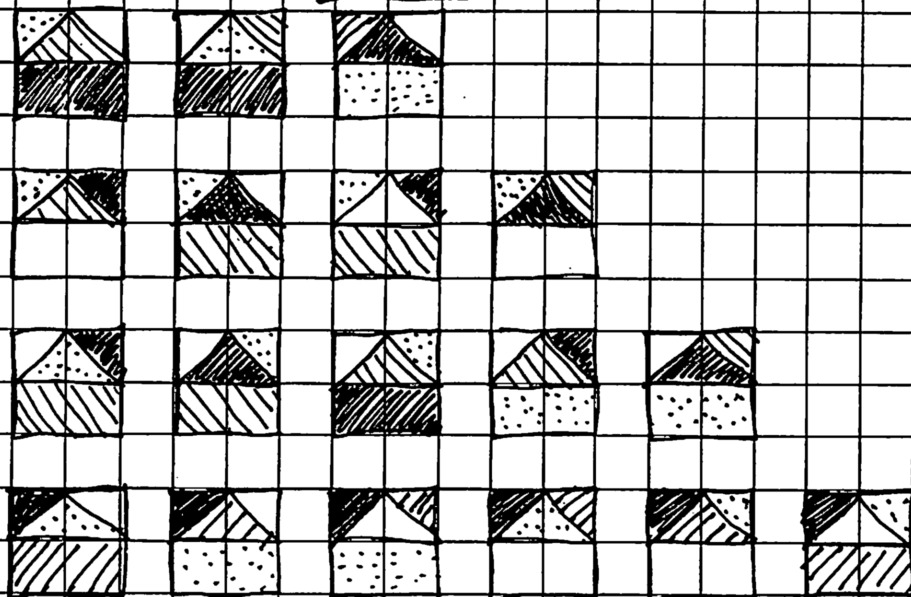
Investigative Processes

The first thing students do when they begin an investigation is to *generate results*. Either from the very beginning, or after they have generated some results randomly, they may *generate results systematically*. From the results generated students will probably *make observations*. They should be encouraged to *look for patterns*. If they are successful in noticing patterns they should be encouraged, if you feel it is appropriate, to *generalise* beyond the original group of results. When looking for patterns or generalising, students should be encouraged to *justify* their statements.



Example of Student's Work Annotated with Processes

Colours



Generating Results

Being Systematic

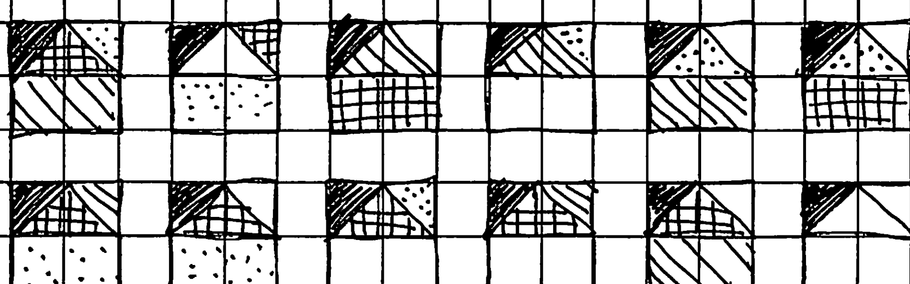
There are six different ways if I keep one the same.

Making an Observation

There are 24 ways altogether.

Making an Observation

5 colours



Generating Results

Being Systematic

There are six ways if I keep 2 the same

Making an Observation

Looking for a Pattern

There are always six ways with 3 colours.

Generalising

There is 2 ways for 2 colours because you can swop them round.

Justifying

Exercise

Write on the following pieces of students' work (*Digit Add* and *Dotty Triangles*), which investigative processes you think are occuring.

Digit Add

I am going to start with 10 and then go up

10 → 9
11 → 11 − 2 → 9
12 → 12 − 3 → 9
13 → 9
14 → 9

Every number goes to 9

46 → 36
47 → 47 − 11 → 36
48 → 48 − 12 → 36
49 → 49 − 13 → 36
45 → 45 − 9 → ~~44~~ 36
44 → 44 − 8 → 36

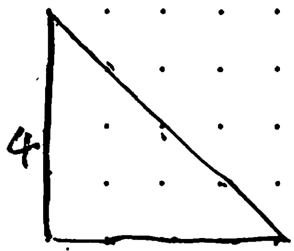
These all go to 36

21 → 18
22 → 18
23 → 18
24 → 18

They will all go to a number in the three times table.

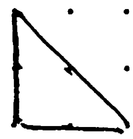
They stay the same because as you add one on you take it off.

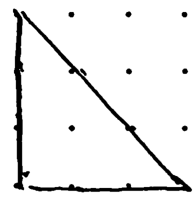
Dotty Triangles



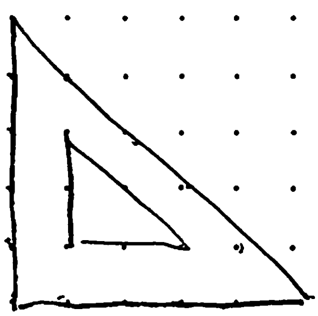
1. 12 dots
2. 3 dots
3. 15 dots
4. 8 sq cm.

3	6	9
0	0	1
3	6	10
$\frac{1}{2}$	2	$4\frac{1}{2}$





The dots on the edge is in the three times table.
The area is half the square



15 The first number is three times the side.
6 That is the same as the little one.
21. That is in the three times table as well.
$12\frac{1}{2}$

Conclusion
1. This number is always in the three times table
2. This number goes up slowly.
3. This number is usually in the three times table
4. This is half the square because half a square is a triangle.

Your First Investigation Lesson

Most school students think maths lessons are about getting work done, which to them means covering pages with completed exercises or getting through workcards or pages in a text book. Some students may believe they are about remembering facts and particular techniques.

The approach in an investigation lesson is different from these, so your students will need to broaden their view of learning.

It is advisable in your first investigation lesson to concentrate on two aspects only:

1. Introducing the investigation to the class

2. The process of generating results

You are advised to keep your first investigation lesson short. Half an hour should be enough time for your introduction and for students to generate results. There should be an opportunity for students who want to continue with the investigation to do so. There should be other work available after half an hour.

When the students are new to investigating you will need to introduce the investigation yourself. You will find that as the students get used to investigative work you will be able to rely increasingly on student material. The students will be able to decide how to pursue the investigation themselves from the initial stimulus.

For the class introduction, as with all good class teaching, try to involve the class as much as possible. Involve *everyone* if you can, not just one or two individuals.
The student activities referred to in this section appear in *Section Three* of the Pack.

Examples

COLOURS (p.82)

You will need coloured chalk.

Make sure the students have the equipment they need, i.e. 4 colours and squared paper.

- Draw the shape shown on p.82 on the board.
- Tell the students to copy it.
- Tell them to use their four colours to colour their shape in.
- While they are doing this, draw three or more of the shape on the board.
- When they have finished, ask three or four students to come up and colour a shape on the blackboard.

The blackboard now shows a few different ways of colouring the shape. You can now pose the question:

'How many different ways can you find?'

DIGIT ADD (p.86)

Ask everyone in the class to:

— write down a two-digit number;
— add the digits together and write down the answer;
— find the difference between the two numbers they have written down.

If students are more familiar with 'subtract' than 'find the difference', then say 'subtract'. Use whatever language will best help your students to get started on the investigation. Ask two students to come up to the board and write down what they have done. (Draw a box on the board so they have somewhere specific to put their work.)

Ask two or three people to give their final answer.

Now tell the students to repeat this for many more two digit numbers. What do they notice?

Your introduction to the investigation should engage the students with the activity. It should not prescribe what they are going to find out.

After your introduction there may be some students who do not understand what to do. This is usually because they are being asked to do something other than what they expect in a maths lesson. It may be that they are used to having a list of instructions to follow. Do not fall into the trap of providing them with a list of instructions if they do not know what to do. Ask them instead to think for themselves.

Student: What do I do?
Teacher: What do you think you do?

Very often the student will give a reply to this which you can agree with. This confirms the student's thinking and will encourage her to trust her own judgement in future.

Sometimes it maybe necessary to think back to an introductory example.

Examples

COLOURS (p.82)

Student: What do I do?
Teacher: What do you think you do?
Student: Colour each space in a different colour?
Teacher: Yes

FLOW-CHART (p.90)

Student: What do I do?
Teacher: What do you think you do?
Student: Do what it says in the flow chart?
Teacher: Yes

Examples continued

DIGIT ADD (p.86)

Student: What do I do?
Teacher: What do you think you do?
Student: I don't know
Teacher: What have you just done?
Student: I've been doing this with numbers. *(Points to work)* Do I do more?
Teacher: Yes

DOTTY TRIANGLES (p.86)

Student: What do I do?
Teacher: What do you think you do?
Student: I don't know
Teacher: What have you just done?
Student: I have done some triangles and counted the dots.
Teacher: So what *could* you do next?
Student: I could do their areas.

When you are not merely a provider of knowledge, but a guide it is no longer necessary to able to predict everything that a student might discover while doing an investigation although you might feel more confident when you start if you are fairly knowledgable about the things which are likely to happen in the particular investigation you are using.

You are still the person 'in charge' in the classroom and it is still your responsibility to define the parameters within which the students work. Those parameters are wider and different from those in a 'chalk and talk' lesson but there may be certain directions which you would rather students did not follow.

✽ Exercise

Choose one of the investigations above to do with a class.

Follow the guidance given on introducing investigations and getting students started.

All the students should now be generating results for the investigation. Remember it is up to you to decide how long this goes on. Do not let the investigation run on unless you feel comfortable with it.

After the lesson

So how did it go?

Write a few notes about what you liked/didn't like about your first investigation lesson. Keep them in your folder with your other papers. You may find them interesting in a few months' time when you might feel differently.

Students Being Systematic

What do we mean by a system?

Being systematic is a fundamental aspect of mathematics.

A system is basically a way of organising things or generating ideas. There are systems for organising books in libraries or albums in a record store. We have systems at home to make it easy to find crockery and cutlery. If you were asked to find how many ways there are drawing two diagonals in a regular nonagon, you would probably begin by drawing in some diagonals and then put them different places in a systematic order. Your system would probably enable you to find the total number.

The ability to be systematic is part of most people's intellectual equipment.

We can encourage the learning of systems by:

—watching carefully what the student is doing and recognising progress over time;
—taking the student's attempted systems and partial systems seriously. We should praise and encourage when students begin to be systematic. It is sometimes tempting to criticise a student if a system seems obvious to us and the student is only partially using it. But the situation is similar to that of children learning to talk; if their talking is always criticised and not taken seriously, they will talk less and learn less quickly.

We can encourage students to improve their ability to be systematic by making sure that there are plenty of contexts where using systems is rewarding. Mathematical investigations provide an ideal opportunity for this.

- The learning of systems can be accelerated by providing examples and demonstrations. This can also be done by peers.
- It is possible to stunt the learning of systems by failing to provide an enviroment which contains them, or by failing to make the use of them sufficiently rewarding.

❇❇❇ Exercise

Choose one of the following investigations:

- *Loopy Numbers* (p.94)
- *Palace Rooms* (p.101)
- *Strips of Squares* (p.109)

Use only the *student material,* not the teacher material, for the investigation.

Start the investigation, working individually.

Stop after about ten minutes, i.e. after everyone has done some work on the investigation, but before anyone is near to a conclusion.

Discuss the *methods* you were using to make progress with the investigation.

Are there systematic ways of generating results for the investigation you chose?

Did everyone use the same system or were there different systems?

The following are systems commonly used by teachers when doing these three investigations: (see *Section Three: Student Activities*, p.81).

LOOPY NUMBERS

* Completing the example shown, then going through starting numbers in order, beginning with 1.

* Having established that there are different loops, trying to find ways that the loops might connect.

* Working backwards: Starting with a number at the bottom of the page and multiplying it by 3, also subtracting 16 from it if it is large enough. This produces a tree-like diagram. It could have loops in it but not necessarily.

PALACE ROOMS

* Generating different results by trial and improvement.

* Choosing one portion of the Venn Diagram and varying the number in that portion systematically.

* Choosing one portion of the Venn Diagram. Finding the maximum and minimum values for that portion which can produce a solution. Using these to count the total possible number of solutions.

STRIPS OF SQUARES

* Starting by keeping the first square shaded and drawing all possible examples (5), then keeping the second square shaded. Continuing this process and checking for repeats.

* Counting results without actually drawing them. Using the same system as above, but mentally.

$$5 + 4 + 3 + 2 + 1 = 15$$

* Applying the formula: ${}_nC_r$

✻ Exercise

Choose a class with which to do one of the three investigations.

Introduce the investigation to the class in the same way as usual and make sure everyone is working at generating results. Look at the methods students are using to generate results.

Can you see any systems or partial systems?

If you see any evidence of students being systematic, even if they are using a system only partially, make some positive comment.

If a student appears to be using a system competently, ask her to explain it to you.

If a student is producing results profusely but randomly, suggest to her that she checks for repeats. This should give encouragement to be systematic.

Making Observations, Looking for Patterns and Generalising

In some investigations it is easy to see a pattern and realise that this is a generalisation which is true in all cases. In others there are patterns and generalisations which are hard to find. There are some investigations which follow clear rules for generating results but the relationship between the variables does not seem to be predictable.

However easy or difficult it is to find a relationship, the following points will help:

- Results generated in an investigation should always be checked before any work is done on trying to find a relationship. There is no point in working from false data.

- If a system is used to generate results it is often easier to see relationships. For instance, in a number investigation if starting numbers are chosen in order, beginning with 1, observing incremental differences in results could lead to the definition of a relationship.

- Observations on results, which could be relevant to the structure of the investigation and therefore to a relationship between variables, should be written down clearly. If observations are to be referred to later they should be distinguishable from rough work.

When students have identified a relationship they should be encouraged to investigate further to see if the relationship is true in general. Ask them to see if it works for other shapes/numbers.

Higher attainers and older students should be encouraged to express relationships algebraically.

❃ Exercise

Choose one of the following investigations to do with a class:

- *Digit Add* (p.86)
- *Dotty Triangles* (p.86)
- *Necklaces* (p.97)
- *Stamp Tearing* (p.107)

In the lesson:

When individual students have sufficient, systematic results ask them *'What do you notice?'*

If students are working towards a generalisation, ask them to try for other shapes/numbers until they are convinced it will always work.

After the lesson:

What patterns or generalisations did students find?

How were these patterns or generalisations expressed? e.g. orally, in writing, in algebra.

Here are some rules which can be found for these investigations. Your students may have found others, or they may have found some of these, but expressed them differently.

The rules are expressed in students' own words.

Digit Add:

- The sum of the digits in the answer is always 9
- The answer is always in the three times table
- The answer is always divisible by 9

Dotty Triangles:

- The number of dots on the perimeter is always in the three times table
- The number of dots inside goes up by one more each time (Triangle Numbers)
- The area is always half a square number

Necklaces:

- It stays the same for two numbers of pearls then goes up one
- If the number of pearls is even you halve it and add one
- If the number of pearls is odd you halve it and add a half

Stamp Tearing:

- The number of ways goes up by one then it comes down by one
- If you are taking less than half the stamps off then you add one to find the number of ways
- If you are taking more than half the stamps off then you subtract one to find the number of ways
- Half the number of stamps gives the maximum number of ways
- The number of ways for half the number of stamps is one more

ⅅ **Exercise**

Choose an investigation which you all find appealing.

Do the investigation together. This may involve working individually some of the time. Concentrate on looking for patterns and generalising. Allow yourselves plenty of time so that everyone has time to find a conclusion.

What patterns or generalisations did people find?
It is quite likely that people will find different patterns for the same investigation.

Are the different patterns related?

Justifying

Justifying used to be considered the preserve of higher mathematics. Justifications which constituted formal proofs were the only ones of value. Now it is recognised that justifying is important for the development of any student's logical reasoning.

Students may spend a lot of time thinking about why something works or why a generalisation seems to be true in all the cases they've looked at. They usually will not write these thoughts down, unless they are high attainers and find a definite and convincing reason.

Encourage all students to write down their ideas about the reasons for things—or even just *thoughts* about something which might impinge on the reason. (See the example of student's work on *Digit Add* with justification, p.86, 'They stay the same because as you add one on you take it off'.) Sometimes there is an obvious reason which most students will be able to see. For instance, in *Dotty Triangles* the number of dots on the perimeter is always a multiple of 3 because the triangle has 3 sides with the same number of dots on each side.

Encouraging and rewarding this kind of thinking will help students to develop the concept of proof more quickly.

In the lesson, when students have discovered or noticed something which they believe to be true, ask them *why* it works. Students new to investigating will probably just tell you that they don't know, but don't be despondent! Encourage them to think about reasons and to write down anything about the investigation which they think is to do with a reason. As they get more practice in thinking about justifications they will get into the habit of writing their ideas down.

❉(❉❉) **Exercise**

Choose an investigation you have worked on before, and for which you already have a generalisation.

Try to justify the generalisation.

Can you find sufficient reasons to justify it?

Can you write a proof for it?

Writing Things Down

When people begin doing investigations they usually do not write anything down. They draw diagrams and jot down numerical results but they do not explain anything. This makes it difficult for someone else to see what is going on. If you cannot see what is going on, you cannot give appropriate guidance.

The following are the sorts of things which it is useful for the student to have written down:

Decisions Made e.g.

- Starting with a particular number
- Working through a set of numbers systematically
- Ignoring a whole group of shapes or numbers because the problem is too complicated
- Changing the question asked to a simpler but related one
- Using a short cut method to arrive at the answer
- Extending the task beyond what was originally asked

The Actual Results

- Shapes and diagrams should be clear but not necessarily neat or pretty
- Numerical **results** should be clearly distinguishable from rough calculations

Observations, Patterns, and Generalisations

These need to be written in clear prose by all students, and/or as algebra by high attainers. If you have a student whose observations and patterns are muddled with other writing, you could suggest she writes a heading, 'What I have noticed', and collect all that is relevant under that heading.

Justifications

Students should write down their ideas about why something works.

Do not ask the students to write *everything* down. This can result in students writing down everything they do:

Example (Digit Add)

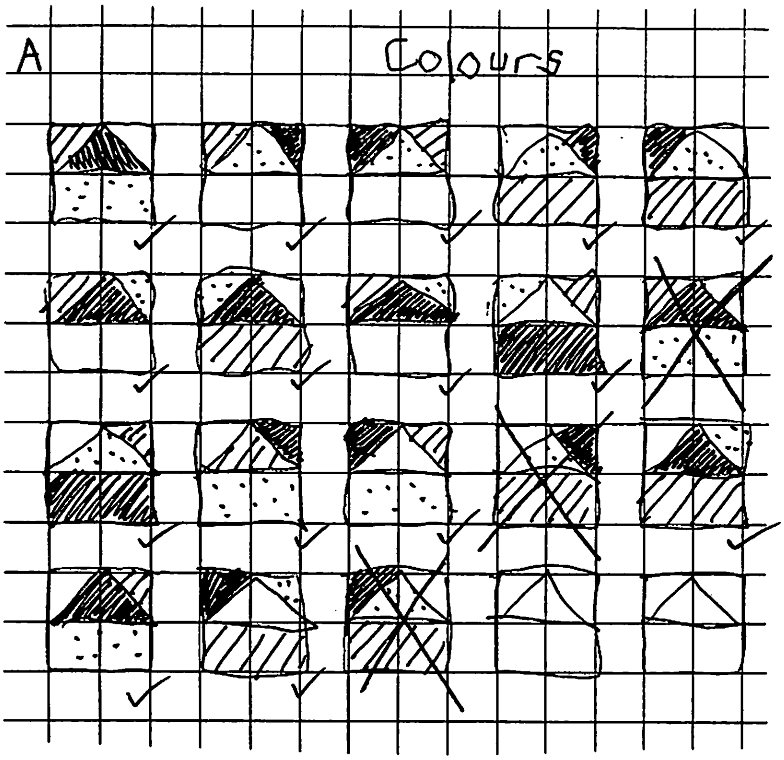
'First I thought of a number 14. Then I added the two numbers together. Then I took 5 from 14 and got 9 . . .'

Students *don't* need to write all this down!

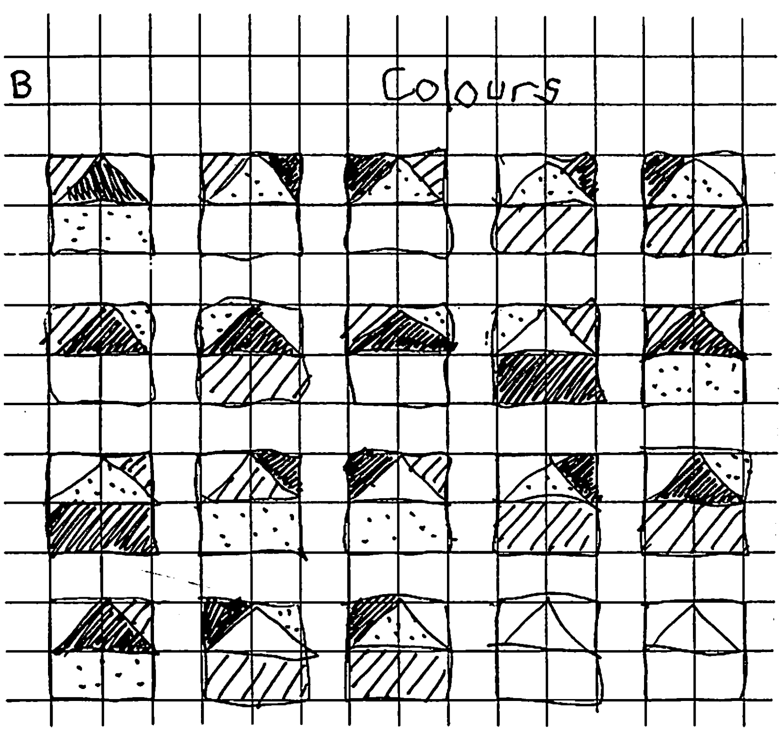
Ways of Marking

✽(✽✽✽✽) Exercise

Here are two identical pieces of students' work. The first is marked badly. The second is marked well. Make a list of the things which you think make the first bad and the second good.



Use a ruler!
Only 11
There are lots more!



Well done.
How many different ways did you find?

Piece A

* There are too many markings on the student's work.

* Repeats have been deleted by the teacher, denying the student the opportunity to find them for herself.

* The critical comment about the ruler is unjustified; the diagrams are clear enough for the purpose.

* The quantity of work is criticised. Either the teacher is criticising the student for not trying hard enough, or the teacher is being negative about the student's ability. To be constructive, comments about effort should be given orally in class, when it is relevant and when the student can remedy the situation. Negative comments about the student's ability won't help.

* The teacher belittles the student's effort.

Piece B

* The student's work is not defaced. It is treated with respect and the student can take pride in her work.

* The student's effort and achievement are praised.

* The student is invited to continue work on the investigation in a way which is challenging but not insulting.

Do and Don't List

DO

— expect students to think of things which you haven't thought of;

— be open to different interpretations of the investigation;

— encourage students to talk to you and to one another about what they are doing;

— encourage students to write down any decisions they make and what they have noticed;

— ask students if they think they have finished;

— let the students decide when they have gone as far as they can/want to;

— praise *effort* as well as mathematical achievement;

— accept work which does not look so pretty but gets the mathematical point across, e.g. in *Colours*, a blob of the colour in each space rather than neat colouring in;

DO *continued*

— give students who want to carry on, *enough* time to complete the investigation to their own satisfaction (this should be lesson time as well as homework);

— respect student's finished pieces of work. Treat them sensitively when you collect them in and look for mathematical achievement in them.

— encourage *students* to look critically at their own work rather than criticise it for them;

— think about what you say to a student who says, 'I've finished it'.

DON'T

— tell students exactly how to do the investigation—they must do mathematics for themselves and not always reproduce the mathematics that they are shown;

— give students starting values in a number investigation—let them choose their own;

— tell students how they should present their results, e.g. do not give them tables or charts to fill in—this will not give them a chance to devise their own systems of presentation;

— be disappointed if your students don't come up with $4 \times 3 \times 2 \times 1$ as a summary of the *Colours* investigation;

— think that the conclusion to an investigation, which is obvious to you, is the only useful thing which can come out of it;

— write your conclusions for the investigation on the board for students to copy in the next lesson;

— make the students, who want to carry on with the investigation, finish it off for homework (by all means let them continue at home, but if they want to spend hours on it they should be allowed lesson time—this is better for you because you can keep track of what they are doing and it is fairer to the students);

— put marks in red pen all over the students' work when you have collected it in;

— skim-read students' work for errors—read what the student has written and make a positive comment.